Math 3236 Statistical Theory

2/7/23

Beta family is conjugated to the Bernoulli.

Experiment with improper prior $B(0, 0)$. I ran a large sample N and at the end I get pN 1 and $(1-p)N$ 0.

$$B(pN, (1-p)N) = \xi(p \mid \underline{x})$$

$$\mathbb{E}(P \mid \underline{x}) = \int p \, \xi(p \mid \underline{x}) \, dp = p$$

$$\mathrm{Var}(P \mid \underline{x}) = \frac{pN(1-p)N}{N^2(N+1)} =$$

$$= \frac{p(1-p)}{N+1}$$

L. L. N. Tells me That

$$\bar{P} \xrightarrow{P} p$$

$$X_N \simeq B(pN, (1-p)N)$$

$$\frac{\sqrt{N}}{\sqrt{p(1-p)}}(X_N - p) \xrightarrow{d} \mathcal{N}(0,1)$$

if $Z \simeq \mathcal{N}(0,1)$

$$X_N \simeq \frac{\sqrt{p(1-p)}}{\sqrt{N}} Z + p$$

p.d.f. of X_N

$$f(x) = x^{pN-1}(1-x)^{(1-p)N-1} \frac{\Gamma(N)}{\Gamma(pN-1)\,\Gamma((1-p)N-1)}$$

$p = \frac{1}{2}$

$$z = \frac{\sqrt{N}}{1/2}\left(X - \frac{1}{2}\right)$$

$$f_Z(z) = \frac{1}{\sqrt{N}} \frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}-1\right)^2} \left(\frac{1}{2} + \frac{z}{2\sqrt{N}}\right)^{\frac{N}{2}-1} \left(\frac{1}{2} - \frac{z}{2\sqrt{N}}\right)^{\frac{N}{2}-1}$$

$$x \to x - \frac{1}{2} \to 2\sqrt{N}\left(x - \frac{1}{2}\right)$$

$$f_Z(z) \xrightarrow[N\to\infty]{} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\left(\frac{1}{2} - \frac{z}{2\sqrt{N}}\right)^{\frac{N}{2}-1} \left(\frac{1}{2} + \frac{z}{2\sqrt{N}}\right)^{\frac{N}{2}-1} =$$

$$\left(\frac{1}{2}\right)^{N-2} \underbrace{\left(1 - \frac{z}{\sqrt{N}}\right)^{\frac{N}{2}-1} \left(1 + \frac{z}{\sqrt{N}}\right)^{N/2+1}}_{h(z)}$$

$$z = 0 \quad h(z) = 1$$

$$h(z) = e^{\underbrace{\left(\frac{N}{2}-1\right)\left(\log\left(1 - \frac{z}{\sqrt{N}}\right) + \log\left(1 + \frac{z}{\sqrt{N}}\right)\right)}_{\psi(z)}}$$

$$z = 0$$

$$\psi''(z) = \left(\frac{1}{1 - \frac{z}{\sqrt{N}}} - \frac{1}{1 + \frac{z}{\sqrt{N}}} \right) \left(\frac{N}{2} - 1 \right)$$

$$\psi'(0) = 0$$

$$\log(1 + x) = x - \frac{x^2}{2} + O(x^3)$$

$$\log\left(1 + \frac{z}{\sqrt{N}}\right) = \frac{z}{\sqrt{N}} - \frac{z^2}{2N} + O\left(\frac{z^3}{N^{3/2}}\right)$$

$$\log\left(1 - \frac{z}{\sqrt{N}}\right) = -\frac{z}{\sqrt{N}} - \frac{z^2}{2N} + O\left(\frac{z^3}{N^{3/2}}\right)$$

$$\psi(z) = \left(\frac{N}{2} - 1\right)\left(-\frac{z^2}{N} + O\left(\frac{z^3}{N^{3/2}}\right)\right)$$

for N large

$$\psi(z) = -\frac{z^2}{2} + \text{smaller stuff.}$$

$$\left(1 - \frac{z}{\sqrt{N}}\right)^{\frac{N}{2} - 1} \left(1 + \frac{z}{\sqrt{N}}\right)^{\frac{N}{2} - 1} \xrightarrow[N \to \infty]{} e^{-\frac{z^2}{2}}$$

$$\left(\frac{1}{2}\right)^{N-2} \frac{1}{\sqrt{N}} \frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}-1\right)^2 \sqrt{N}} \to \frac{1}{\sqrt{2\pi}}$$

Stirling formula ad nauseam!

if X_N is $B\left(\frac{N}{2}, \frac{N}{2}\right)$

$$2\sqrt{N}\left(X_N - \frac{1}{2}\right) \simeq N(0, 1)$$

$$X_N \simeq N\left(\frac{1}{2}, \frac{1}{4N}\right)$$

Bayes Estimators.

Cost function

$L(\theta, a)$

$L(\theta, a)$ is the cost you pay if the true value is θ and you think it is a.

$$\min_{a} \int L(\theta, a)\, \xi(\theta \mid \underline{x})\, d\theta$$

$$\arg\min \mathbb{E}_{\theta}\left(L(\theta, a) \mid \underline{x} \right) = \delta(\underline{x})$$

$\delta(\underline{x})$ is a statistics.

Bayes estimator for the cost function L.

$$L(\theta, a) = (\theta - a)^2$$

$$\delta(\underline{x}) = \mathbb{E}(\theta \mid \underline{x})$$

---

x_i are Bernoulli

θ has a prior that is

$B(\alpha, \beta)$

After N observation with

n positive result $n = \sum_i x_i$

$$\delta(\underline{x}) = \frac{\alpha + \sum_i x_i}{\alpha + \beta + N}$$

$$\delta(\underline{X}) = \frac{\alpha + \sum_i X_i}{\alpha + \beta + N}$$

Since α, β are finite

if X_i are i.i.d with par p.

$$\delta(\underline{X}) \xrightarrow{P} p$$

$l\theta$-al

$$\delta(\underline{X}) = \text{median } \xi(\theta \mid x)$$
